Indian J. Phys. **70B** (6), 465–474 (1996)

IJP B
— an international journal

General expression for the generating function of special functions in mathematical physics

M A Jafarizadeh[(A,B,1)] and H Fakhri[(A)]

A) Department of Physics, Tabriz University, Tabriz-51664, Iran

B) Institute for Studies in Theoretical Physics and Mathematics, Tehran-19395-1795, Iran

Received 18 April 1996, accepted 21 August 1996

Abstract : Using Lagranges expansion formula and generalized Rodrigues formula, we give a general expression for the generating function of the special functions in mathematical physics in a very elegant method Also, we introduced two new orthogonal polynomials with their corresponding second order differential equations

Keywords : Special functions, generating functions, Lagranges expansion

PACS Nos. : 02 30 Gp, 02 90 +p

1. Introduction

The special functions and in particular, the orthogonal polynomials appear almost in every branch of applied mathematics, physics, engineering and recently, they have been playing an important role in 2d-gravity [1], high energy physics [2] and numerical solutions of differential equations [3]. Orthogonal polynomials of order n denoted by $\Phi_n(x)$ are defined through a weight function which are non-negative in a given interval (a,b), as

$$\int_a^b W(x)\Phi_n(x)\Phi_m(x)dx = \delta_{nm}.$$

Therefore, we can have as many orthogonal polynomials as we wish, but there are very few ones which satisfy a given second order linear differential equations. These are the orthogonal polynomials which can be constructed through a master function which is at most a second order polynomial [4].

In Section 2, for the sake of completeness, we explain first how one can obtain the classical orthogonal polynomials from a given master function. In Section 3, we define two rather new orthogonal polynomials together with their corresponding differential equations.

1-E-mail : MAJAFA @ VAX.IPM.AC.IR

To our knowledge, this is the first time they being introduced in literature. In Section 4, using Lagranges expansion formula [5], we give a general expression for the generating function of the classical orthogonal polynomial which is actually the main result of this paper. Finally, in Section 5, we give the generating functions and differential equations corresponding to special functions associated with the classical orthogonal polynomials introduced in Section 2. The paper is ended with a brief conclusion, and an appendix concerning the derivation of Lagranges expansion formula.

2. Master function and classical orthogonal polynomials

By introducing the polynomial $A(x)$ which is at most a second order polynomial, one can obtain the main ingredient of orthogonal polynomial, *i.e.* the non-negative weight function $W(x)$ and the interval (a,b) as follows. First, the non-negative weight function $W(x)$ is defined such that $\frac{1}{W(x)}\frac{d}{dx}(W(x)A(x))$ is at most a first order polynomial and interval (a,b) is chosen in such a way that $W(a)A(a) = W(b)A(b) = 0$. Then one can define the second order linear operator $L = \frac{1}{W(x)}\frac{d}{dx}\left(W(x)A(x)\frac{d}{dx}\right)$ with the following interesting properties :

(1) L is a self-adjoint linear operator.

(2) L transforms, a given polynomial of order m to another polynomial of order m at most.

(3) The expression $\frac{1}{W(x)}\left(\frac{d}{dx}\right)^n (A^n(x)W(x))$ is a polynomial of order at most n, which is indeed Rodrigues formula for the classical orthogonal polynomials

(4) The polynomials

$$\Phi_n(x) = \frac{a_n}{W(x)}\left(\frac{d}{dx}\right)^n (A^n(x)W(x)) \tag{1}$$

are orthogonal with respect to the weight function $W(x)$ in interval (a,b), as defined above and one can find the a_n's simply by comparing the coefficient of highest power of Φ_n's with those of the traditionally defined special orthogonal polynomials.

(5) The polynomials $\Phi_n(x)$ are eigenfunctions of operator L and therefore, satisfy a second order linear differential equation

$$\frac{1}{W(x)}\frac{d}{dx}\left(W(x)A(x)\frac{d}{dx}\Phi_n(x)\right) = -\gamma_n\Phi_n(x). \tag{2}$$

In order to have polynomial solution of degree n in differential equation (2), the γ_n's must be given by

$$\gamma_n = -n\left(\frac{(W(x)A(x))'}{W(x)}\right)' - \frac{n(n-1)}{2}A''(x),$$

where in the derivation of γ_n's we have compared simply the coefficient of the highest power, *i.e.* x^n in both sides of (2), and we have taken into account the fact that $A(x)$ and

$\frac{1}{W(x)}(W(x)A(x))'$ are polynomials of order at most two and one, respectively. Thus, general form of the differential equation is as follows

$$A(x)\Phi_n''(x) + \frac{(W(x)A(x))'}{W(x)}\Phi_n'(x) - \left[n\left(\frac{(W(x)A(x))'}{W(x)}\right)' + \frac{n(n-1)}{2}A''(x)\right]\Phi_n(x) = 0. \tag{3}$$

The Table 1 contains the standard classical differential equations and their corresponding Rodrigues formula.

3. Two new orthogonal polynomials

Here, we introduce two new differential equations for the first time simply by choosing $A(x) = x^2$ and $A(x) = 1 + x^2$ respectively, up to rescaling and translation of variable x.

First we choose $A(x) = x^2$, then the condition on $\frac{1}{W(x)}(x^2W(x))'$ that is, to be first order polynomial (at most), will lead to the following first order differential equation for $W(x)$

$$\frac{W'(x)}{W(x)} = \frac{\alpha x+1}{x^2},$$

with solution $w(x) = x^{\alpha}e^{-1/x}$. Then $W(x)A(x) = x^{\alpha+2}\exp\left(\frac{-1}{x}\right)$ will vanish only at points $x = 0$ and $x \Rightarrow \infty$ if $\alpha < -2$. Now putting $A(x) = x^2$ and $W(x) = x^{\alpha}\exp(-1/x)$ in general differential equation (3), we get the following differential equation

$$x^2F_n''^{(\alpha)}(x) + [1 + (\alpha+2)x]F_n'^{(\alpha)}(x) - n(x+\alpha+1)F_n^{(\alpha)}(x) = 0.$$

Similarly by choosing $A(x) = 1 + x^2$ and requiring that $\frac{1}{W(x)}((1+x^2)W(x))'$ be at most first order, will lead to the following first order differential equation

$$\frac{W'(x)}{W(x)} = \frac{2\alpha x+\beta}{1+x^2},$$

with solution $W(x) = (1+x^2)^{\alpha}\exp(\beta\,\mathrm{Arctg}x)$. Then $A(x)W(x) = (1+x^2)^{\alpha+1}\exp(\beta\,\mathrm{Arctg}x)$ will vanish only at $x \Rightarrow \pm\infty$ provided that $\alpha < -1$, $\beta \in \mathrm{IR}$. Again by putting $A(x) = 1 + x^2$ and $W(x) = (1+x^2)^{\alpha}\exp(\beta\,\mathrm{Arctg}x)$ in (3), we get the following differential equation

$$(1+x^2)J_n''^{(\alpha,\beta)}(x) + [\beta + 2(\alpha+1)x]J_n'^{(\alpha,\beta)}(x) - n(n+2\alpha+1)J_n^{(\alpha,\beta)}(x) = 0.$$

The Rodrigues formula and differential equations are given in Table 2.

Table 1. Classical differential equations and Rodrigues formula.

Polynomials	$A(x)$	$W(x)$	Interval	Rodrigues formula and differential equation
Laguerre	x	$e^{-x}x^{\beta}$	$[0, +\infty]$	$L_n^{(\beta)}(x) = \frac{1}{n!} e^{x} x^{-\beta} \frac{d^n}{dx^n}\left(e^{-x}x^{\beta+n}\right)$
		$\beta > -1$		$xL_n''^{(\beta)}(x) + (\beta+1-x)L_n'^{(\beta)}(x) + nL_n^{(\beta)}(x) = 0$
Jacobi	$1-x^2$	$(1-x)^{\alpha}(1+x)^{\beta}$	$[-1, +1]$	$P_n^{(\alpha,\beta)} = \frac{(-1)^n}{2^n n! (1-x)^{\alpha}(1+x)^{\beta}} \frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$
		$\alpha, \beta > -1$		$(1-x^2)P_n''^{(\alpha,\beta)}(x) + \left[(\beta-\alpha)-(\alpha+\beta+2)x\right]P_n'^{(\alpha,\beta)}(x) + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = 0$
Hermite	1	$e^{-x^2/2}$	$[-\infty, +\infty]$	$H_n(x) = (-1)^n \exp\left(x^2/2\right) \frac{d^n}{dx^n} \exp\left(-x^2/2\right)$
				$H_n''(x) - xH_n'(x) + nH_n(x) = 0$
Hyper-geometric	$x(1-x)$	$x^{\alpha}(1-x)^{\beta}$	$[0, +1]$	$F_n^{(\alpha,\beta)}(x) = \frac{1}{n!} x^{-\alpha}(1-x)^{-\beta} \frac{d^n}{dx^n}\left[x^{\alpha+n}(1-x)^{\beta+n}\right]$
		$\alpha, \beta > -1$		$x(1-x)F_n''^{(\alpha,\beta)}(x) + \left[(\alpha+1)-(\alpha+\beta+2)x\right]F_n'^{(\alpha,\beta)}(x) + n(n+\alpha+\beta+1)F_n^{(\alpha,\beta)}(x) = 0$

Table 2. Differential equations and Rodrigues formula.

Polynomials	$A(x)$	$W(x)$	Interval	Rodrigues formula and differential equation
	x^2	$x^{\alpha} e^{-1/x}$	$[0, \infty)$	$F_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp(1/x) \frac{d^n}{dx^n}\left[x^{\alpha+2n} \exp(-1/x)\right]$
		$\alpha < -2$		$x^2 F_n^{\prime\prime(\alpha)}(x) + [1+(\alpha+2)x] F_n^{\prime(\alpha)}(x) - n(n+\alpha+1) F_n^{(\alpha)}(x) = 0$
	$1+x^2$	$(1+x^2)^{\alpha} \exp(\beta \operatorname{Arctg} x)$	$(-\infty, +\infty)$	$J_n^{(\alpha,\beta)}(x) = \frac{1}{n!}(1+x^2)^{-\alpha} \exp(-\beta \operatorname{Arctg} x) \frac{d^n}{dx^n}\left[(1+x^2)^{\alpha+n} \exp(\beta \operatorname{Arctg} x)\right]$
		$\alpha < -1$		$(1+x^2) J_n^{\prime\prime(\alpha,\beta)}(x) + [\beta + 2(\alpha+1)x] J_n^{\prime(\alpha,\beta)}(x)$
		$\beta \in IR$		$- n(n + 2\alpha + 1) J_n^{(\alpha,\beta)}(x) = 0$

What makes them different from the classical one is their finite number. This number depends on the parameter of their corresponding weight function given in Table 2. Actually m, the number of orthogonal polynomials, is less than or equal to $-(\alpha+1)/2$ for the first one and also is less than or equal to $-\alpha-1/2$ for the second one, respectively.

4. Generating function of classical orthogonal polynomials

Using Langranges expansion formula for an analytic function $f(z)$ in and on the simple contour C [5], that is

$$f(z) = f(x) + \sum^{\infty} \frac{t^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left\{f'(x)(\phi(x))^n\right\},$$

where $z = x + t\phi(z)$ and $\phi(z)$ being an analytic function in and on the same contour C, satisfying

$$|t\phi(\zeta)| < |\zeta - x| \tag{4}$$

for $\zeta \in C$ and x inside C, and by choosing $\frac{d}{dz}f(z) = W(z)$ and $\phi(x) = A(x)$ we have

$$\left(\frac{W(z)}{W(x)}\frac{dz}{dx}\right)_{z=z_0} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{1}{W(x)}\frac{d^n}{dx^n}\left(A^n(x)W(x)\right). \tag{5}$$

Since $A(z)$ is at most second order polynomial of z, therefore the equation of $z = x + tA(z)$ have at most two roots, z_0 is one of two roots and it is choosen such that it becomes equal to x as t equal to zero. In this time with using (1), we get

$$\left(\frac{W(z)}{W(x)}\frac{dz}{dx}\right)_{z=z_0} = \sum_{n=0}^{\infty} \frac{t^n}{n!a_n}\Phi_n(x). \tag{6}$$

From equation of $z = x + tA(z)$ we have $\left(\frac{dz}{dx}\right)_{z=z_0} = \frac{1}{1-tA'(z_0)}$, therefore we can also write the expression $\left(\frac{W(z)}{W(x)}\frac{dz}{dx}\right)_{z=z_0}$ in the following form

$$\left(\frac{W(z)}{W(x)}\frac{dz}{dx}\right)_{z=z_0} = \frac{1}{W(x)}\int_{a'}^{b'} dy\, W(y)\,\delta(y-x-tA(y)), \tag{7}$$

where the a' and b' are choosen in such a way that the interval (a', b') contains only the root z_0.

Now using this new expression from our general generating function, we can rather easily show that it generates the set of orthogonal polynomials $\{\Phi_n\}$ having $W(x)$

as the weight function. In order to prove this, we multiply our generating function (7) by $\Phi_m(x)\,W(x)$ and then integrating it over interval (a, b), get

$$\int_a^b \left(\frac{W(z)}{W(x)}\frac{dz}{dx}\right)_{z_0} \Phi_m(x)\,W(x)dx = \int_a^b dx \int_{a'}^{b'} dy\,W(y)\delta(y-x-tA(y))\,\Phi_m(x)$$

$$= \int_{a'}^{b'} dy\,W(y)\,\Phi_m(y-tA(y)), \tag{8}$$

where we have integrated over x using the Dirac-delta function. On the other hand using the equation (6) and exchange of sum and integral we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!a_n}\int_a^b \Phi_n(x)\Phi_m(x)\,W(x)dx = \int_{a'}^{b'} dy\,W(y)\,\Phi_m(y-tA(y)). \tag{9}$$

Now since $\Phi_m(x)$ is polynomial of order m, then the right-hand side of eq. (9) is a polynomial of order m in term of t. Therefore by comparing both sides of expression (9), we have

$$\int_a^b \Phi_n(x)\Phi_m(x)\,W(x)dx = 0 \qquad \text{for } n > m,$$

which proves the orthogonality of set $\{\Phi_n\}$.

We come to the main result of this paper, where the left hand side of formula (6) is the general expression for the generating function of classical orthogonal polynomials.

Table 3. Generating functions of classical orthogonal polynomials

Polynomials	$A(x)$	$W(x)$	Generating function
Laguerre	x	$e^{-x}x^{\beta}$, $\beta > -1$	$\sum_{n=0}^{\infty} t^n L_n^{(\beta)}(x) = \frac{\exp\left(\frac{-xt}{1-t}\right)}{(1-t)^{\beta+1}}$
Jacobi	$1-x^2$	$(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta > -1$	$\sum_{n=0}^{\infty} t^n P_n^{(\alpha,\beta)}(x) = \frac{(1-z_0)^{\alpha}(1+z_0)^{\beta}}{(1-x)^{\alpha}(1+x)^{\beta}}\frac{1}{(1-2xt+t^2)^{1/2}}$, $z_0 = \frac{1}{t} - \frac{1}{t}\left(1-2xt+t^2\right)^{1/2}$
Hermite	1	$e^{-x^2/2}$	$\sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x) = \exp\left(\frac{-t^2+2xt}{2}\right)$
Hyper-geometric	$x(1-x)$	$x^{\alpha}(1-x)^{\beta}$, $\alpha,\beta > -1$	$\sum_{n=0}^{\infty} t^n F_n^{(\alpha,\beta)}(x) = -\frac{z_0^{\alpha}(1-z_0)^{\beta}}{x^{\alpha}(1-x)^{\beta}}\frac{1}{\left((1-t)^2+4xt\right)^{1/2}}$, $z_0 = \frac{-(1-t)}{2t} - \frac{1}{2t}\left((1-t)^2+4xt\right)^{1/2}$

Table 3. *(Cont'd.).*

Polynomials	$A(x)$	$W(x)$	Generating function
	x^2	$x^\alpha e^{-1/x}$ $\alpha < -2$	$\sum_{n=0}^{\infty} t^n F_n^{(\alpha)}(x) = \frac{z_0^\alpha \exp(-1/z_0)}{x^\alpha \exp(-1/x)} \frac{1}{(1-4xt)^{1/2}}$ $z_0 = \frac{1}{2t} - \frac{1}{2t}(1-4xt)^{1/2}$
	$1+x^2$	$(1+x^2)^\alpha$ $\exp(\beta \operatorname{Arctg} x)$ $\alpha < -1$ $\beta \in IR$	$\sum_{n=0}^{\infty} t^n J_n^{(\alpha,\beta)}(x) = \frac{(1+z_0^2)^\alpha \exp(\beta \operatorname{Arctg} z_0)}{(1+x^2)^\alpha \exp(\beta \operatorname{Arctg} x)}$ $\times \frac{1}{(1-4t^2-4xt)^{1/2}}$ $z_0 = \frac{1}{2t} - \frac{1}{2t}(1-4t^2-4xt)^{1/2}$

We give in Table 3 corresponding generating functions and related interval of parameter t which can be obtained from correlation (4).

5. Associated special functions

One can easily associate a special function to the classical orthogonal polynomial just by differentiating the polynomial $\Phi_n(x)$ m times, then multiplying it by $(-1)^m\ (A(x))^{m/2}$, that m is n at most n. The resulting associated special function

$$\Phi_{n,m}(x) = (-1)^m (A(x))^{m/2} \frac{d^m}{dx^m} \Phi_n(x)$$

will satisfy the following differential equation.

$$A(x)\Phi''_{n,m}(x) + \frac{(W(x)A(x))'}{W(x)} \Phi'_{n,m}(x) + \left[\frac{1}{2}(m^2 - n^2 - 2m + n)A''(x) + (m-n)\left(\frac{(W(x)A(x))'}{W(x)}\right)' - \frac{m^2}{4}\frac{A'^2(x)}{A(x)} - \frac{m}{2}\frac{W'(x)A'(x)}{W(x)}\right]\Phi_{n,m}(x) = 0. \tag{10}$$

For Hermite and Laguerre polynomials, the above procedure does not give a new result, that is associated Hermite and Laguerre functions are again of the kind of Hermite and Laguerre polynomials. But in other cases, the corresponding-associated functions are of different forms than the polynomials themselves. The related generating function can be obtained from (6) simply by taking derivative of both sides m times and multiplying it by $(A(x))^{m/2}$. This will lead to the following result

$$(-1)^m (A(x))^{m/2} \left(\frac{d}{dx}\right)^m \left(\frac{W(z)}{W(x)} \frac{dz}{dx}\right) = \sum_{n=0}^{\infty} \frac{t^n}{a_n n!} \Phi_{n,m}(x).$$

For example if we choose $A(x) = x(1-x)$, then for the generating function of associated hypergeometric function with $\alpha = \beta = 0$, we get

$$\sum_{n=0} t^n F_{n,m}^{(0,0)}(x) = (2m-1)!!\,(x(1-x))^{m/2}\,(-2t)^m\,\left((1-t)^2 + 4xt\right)^{-m-1/2}.$$

6. Conclusion

By using the master function [4], we introduced two new important orthogonal polynomials and then by combination of generalized Rodrigues and Lagranges expansion formula, we could give a general expression for the generating function of the classical orthogonal polynomial, whereas other methods of deriving the generating function is very tedious and also not unique; that is one should derive the corresponding generating function for each polynomial separately. Generating functions can play an important role in obtaining the asymptotic expansion of orthogonal polynomials for large n [6].

Finally, we have also given general expression for the generating functions of the associated special functions which can be very useful in studying their asymptotic expansion for large n.

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Appendix

Lagranges expansion formula

In derivation of the Lagranges expansion formula, one needs first the famous theorem about the zeros of holomorphic function $\psi(z)$. If $\psi(z)$ and $\phi(z)$ are holomorphic functions inside a simple contour C, then we will have

$$\frac{1}{2\pi i}\oint_C \phi(z)\,\frac{\psi'(z)}{\psi(z)}\,dz = \sum_k n_k\,\phi(z_k), \tag{A-1}$$

where n_k's are the orders of zeros z_k inside C. It can be proved simply using Cauchy's theorem. Actually, the contour C can be deformed to new contours around zeros of $\psi(z)$ inside C. Near each zero z_k we assume

$$\psi(z) = (z-z_k)^{n_k}\,\psi_k(z), \qquad \psi_k(z_k) \neq 0;$$

thus,
$$\frac{\psi'(z)}{\psi(z)} = \frac{n_k}{z-z_k} + \frac{\psi_k'(z)}{\psi_k(z)}.$$

Obviously, $\frac{\psi_k'(z)}{\psi_k(z)}$ is analytic in z_k and (A-1) follows; let $\phi(z) := 1$, then we have

$$\frac{1}{2\pi i}\oint_C \frac{\psi'(z)}{\psi(z)}dz = \sum_k n_k . \tag{A-2}$$

Now, let $f(z)$ and $\phi(z)$ are two analytic functions in and on the C, and $z_0 = x$ is a point within C. If for points ζ on C, the parameter t satisfies

$$\left|t\phi(\zeta)\right| < \left|\zeta - x\right|, \tag{A-3}$$

then equation of $z = x + t\phi(z)$ has one and only one root inside C. Because, applying (A-2) to the function $\psi(z) := z - x - t\phi(z)$ and taking note of condition (A-3), we have

$$\sum n_k = \frac{1}{2\pi i}\oint_C \frac{1-t\phi'(\zeta)}{\zeta - x - t\phi(\zeta)}d\zeta = \frac{1}{2\pi i}\oint_C \left[1-t\phi'(\zeta)\right]\sum_{n=0}^{\infty}\frac{\left(t\phi(\zeta)\right)^n}{\left(\zeta-x\right)^{n+1}}d\zeta$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d^n}{dx^n}\left[\phi(x)\right]^n - \sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}\frac{d^{n+1}}{dx^{n+1}}\left(\phi(x)\right)^{n+1} = 1.$$

Therefore, $\frac{\psi'(z)}{\psi(z)}$ has a simple pole and then $\psi(z) = 0$ has one and only one root inside C.

If z be the only root of $z = x + t\phi(z)$ in C, then by (A-1), we have

$$\frac{1}{2\pi i}\oint_C f(\zeta)\frac{\psi'(\zeta)}{\psi(\zeta)}d\zeta = f(z).$$

Then,
$$f(z) = \frac{1}{2\pi i}\oint_C f(\zeta)\frac{\psi'(\zeta)}{\psi(\zeta)}d\zeta = \frac{1}{2\pi i}\oint_C f(\zeta)\frac{1-t\phi'(\zeta)}{\zeta - x - t\phi(\zeta)}d\zeta$$

$$= \frac{1}{2\pi i}\oint_C f(\zeta)\left[1-t\phi'(\zeta)\right]\sum_{n=0}^{\infty} t^n\frac{\left(\phi(\zeta)\right)^n}{(\zeta-x)^{n+1}}d\zeta$$

$$= f(x) + \sum^{\infty}\frac{t^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(f'(x)\left(\phi(x)\right)^n\right).$$